" RIEMANNIAN GEOMETRY AND THE EQUATIONS OF MOTION "

by

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INTRODUCTION

When Einstein first introduced the notion of the geodesic of a particle in the General Theory of Relativity (1), he did so through the principle of equivalence, postulating that the motion of a particle under the influence of a gravitational field must be indistinguishable locally from inertial motion. The geodesic equations contain the Christoffel symbols of the second kind, field variables which effectively take the place of the force of classical theory. The question arises, whether these equations can properly be called equations of motion, if the Christoffel symbols are singular at the location of the particle; for this is the case with the only solution of the field equations in empty space which is static, has spherical symmetry, and which goes over into the flat metric at infinity, i.e. the Schwartzschild solution.

In 1938, Einstein, Infeld, and Hoffmann (2,3) collaborated in a series of papers in which the necessity of considering the above question was obviated. They succeeded in deriving the equations of motion of a particle in a gravitational field directly from the field equations, without bringing in the idea of a geodesic at all.

In this paper, an investigation is made of a more general set of field equations, derived from a Hamiltonian principle using an integrand slightly more complicated than that which gives the accepted

equations. This set, on the basis of the principle of simplicity, has not been exhaustively examined previously, for it involves fourth derivatives of the metric tensor. However, it will be shown that these new equations reduce to the original ones as a certain parameter tends to zero, and moreover give, for the static case, a solution which is regular at the location of the particle. So we find that the equations of motion, which are geodesics, follow directly from these field equations.

THEORY

We are looking for a symmetric second-order tensor, $L^{\mu\nu}$, depending only on the metric tensor and its derivatives, which we can identify with the stress-energy tensor $T^{\mu\nu}$. The principle of conservation of energy-momentum states, however, that $T^{\mu\nu}{}_{;\nu}=0$. So our second-order tensor must satisfy $L^{\mu\nu}{}_{;\nu}=0$ where ; signifies covariant differentiation. (The notation , is used to denote regular differentiation).

Following Schrödinger (4), we show in Appendix A that if $\mathfrak{M}$ is any scalar density that depends only on the g_{ik} and their derivatives with respect to the coordinates up to any finite order, then the invariant divergence vanishes identically for the tensor density that is constituted by the Hamiltonian derivatives of $\mathfrak{M}$; i.e., let $I=\int \mathfrak{M}\, dx^4$. Compute δI and perform the partial integrations needed to put it into the form $\delta I = \int (\delta \mathfrak{M}/\delta g_{ik})\, \delta g_{ik}\, dx^4$. Let $\mathfrak{M}^{ik} = \delta\mathfrak{M}/\delta g_{ik}$. Then $\mathfrak{M}^{ik}{}_{;i}=0$, or, $(\mathfrak{M}^{ik}/\sqrt{-g})_{;}=0$. So we take $L^{\mu\nu}$ equal to $\mathfrak{M}^{\mu\nu}/\sqrt{-g}$ and identify this with $T^{\mu\nu}$.

We shall wish to consider, as the integrand in our Hamiltonian principle, the product of $\sqrt{-g}$ with F, where F is a scalar depending only on g^{ik} and R_{ik} as independent variables (R_{ik} is the Einstein-tensor). Using the notation, $\beta^{ik} = \delta F/\delta R_{ik}$, we find, (Appendix B)

$$L^{ik}{}_{;i} = \{ (g^{\alpha\epsilon}\beta^{ik} + g^{ki}\beta^{\epsilon\alpha} - g^{\alpha i}\beta^{\epsilon k} - g^{\alpha k}\beta^{i\epsilon})_{;\alpha\epsilon} + Fg^{ik} + 2\delta F/\delta g_{ik} \}_{;i} = 0 \qquad (1)$$

Therefore, following our previous analysis, $L^{ik} = \eta T^{ik}$ where L^{ik} is the expression in parenthesis, and η is a constant to be determined.

We are now ready to consider a definite form for F. Instead of taking $F = R\sqrt{-g}$, as Einstein did, we consider the next simplest expression, $F = R + aR^2 + bR_{ij}R^{ij}$, where R is the curvature scalar, and a, b are arbitrary parameters. Remembering that $R = g^{ij}R_{ij}$ where R_{ij} and g^{ij} are our independent variables, and that $\delta g^{\alpha\beta}/\delta g_{ik} = -g^{i\alpha}g^{k\beta}$, we have

$$\frac{\delta R}{\delta R_{ik}} = g^{ik} \qquad \frac{\delta R^2}{\delta R_{ik}} = 2Rg^{ik} \qquad \frac{\delta R_{lj}R^{lj}}{\delta R_{ik}} = 2R^{ik}$$

$$\frac{\delta R}{\delta g_{ik}} = -R_{\alpha\beta}g^{\alpha i}g^{\beta k} \qquad \frac{\delta R^2}{\delta g_{ik}} = -2RR^{ik} \qquad \frac{\delta R_{lj}R^{lj}}{\delta g_{ik}} = -R^{\alpha k}R_{\alpha\beta}g^{i\beta}$$

Substituting into equation (1) we get (Appendix C),

$$Q_{pq}(R) + a\,Q_{pq}(R^2) + b\,Q_{pq}(R_{ij}R^{ij}) = \eta g_{ip}g_{kq}T^{ik}$$
$$= \eta T_{pq} \qquad (2)$$

where
$$Q_{pq}(R) = -2\left(R_{pq} - \tfrac{1}{2} g_{pq} R\right)$$
$$Q_{pq}(R^2) = 2\left(g^{ik}g^{\alpha\epsilon} + g^{\epsilon\alpha}g^{ki} - g^{\epsilon k}g^{\alpha i} - g^{i\epsilon}g^{\alpha k}\right)R_{;\alpha\epsilon} + R^2 g^{ik} - 4RR^{ik}$$
$$Q_{pq}(R_{ij}R^{ij}) = 2\left(g^{\alpha\epsilon}\delta_p^{\mu}\delta_q^{\nu} + g_{pq}g^{\mu\epsilon}g^{\nu\alpha} - \delta_p^{\alpha}\delta_q^{\nu}g^{\epsilon\mu} - g^{\nu\epsilon}\delta_q^{\alpha}\delta_p^{\mu}\right)R_{\mu\nu;\alpha\epsilon} + R_{\mu\nu}R^{\mu\nu}g_{pq} - 2R^{\alpha k}R_{\alpha p}g_{kq}.$$

Now assume that the gravitational fields (i.e., the deviations of the actual metric from the flat metric) encountered in nature are so weak that the non-linear character of the field equations leads only to secondary effects. So we apply the linear approximation:

$$g_{\rho\sigma} = \epsilon_{\rho\sigma} + \lambda \, \underset{1}{h}_{\rho\sigma} + \lambda^2 \, \underset{2}{h}_{\rho\sigma} + \dots$$

where λ is the parameter of expansion and a small constant, $\epsilon_{\rho\sigma}$ is the flat metric $\begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & +1 \end{pmatrix}$ where we have taken $c^2 = 1$.

To first order in λ in a system where $\sigma_{\mu} = 0$, we calculate (Appendix D),

$$Q_{pq}(R) = -\lambda \epsilon^{\alpha\beta} \gamma_{pq,\alpha\beta}$$

$$Q_{pq}(R^2) = -2\lambda \epsilon_{pq} \epsilon^{\alpha\epsilon} \epsilon^{\alpha\beta} \gamma_{,\alpha\beta\alpha\epsilon} + 2\lambda \epsilon^{\alpha\beta} \gamma_{,\alpha\beta pq}$$

$$Q_{pq}(R_{ij}R^{ij}) = \lambda [\epsilon^{\alpha\epsilon} \epsilon^{\alpha\beta} \gamma_{pq,\alpha\beta\alpha\epsilon} - \epsilon_{pq} \epsilon^{\alpha\beta} \epsilon^{\alpha\epsilon} \gamma_{,\alpha\beta\alpha\epsilon}$$
$$+ \epsilon^{\alpha\beta} \gamma_{,\alpha\beta pq}]$$

where $\sigma_\mu = \epsilon^{\alpha\beta} \gamma_{\mu\alpha,\beta}$

and $\gamma_{\mu\nu}$ is defined by $h_{\mu\nu} = \gamma_{\mu\nu} - \frac{1}{2}\epsilon_{\mu\nu}\gamma$

$$\gamma = -h = -\epsilon^{\mu\nu} h_{\mu\nu}$$

Substituting into equation (2), we get

$$-\lambda \epsilon^{\alpha\beta} \gamma_{pq,\alpha\beta} + 2a\lambda (\epsilon^{\alpha\beta} \gamma_{,\alpha\beta pq} - \epsilon_{pq} \epsilon^{\alpha\epsilon} \epsilon^{\alpha\beta} \gamma_{,\alpha\beta\alpha\epsilon})$$
$$+ b\lambda (\epsilon^{\alpha\beta} \gamma_{,\alpha\beta pq} - \epsilon_{pq} \epsilon^{\alpha\epsilon} \epsilon^{\alpha\beta} \gamma_{,\alpha\beta\alpha\epsilon}$$
$$+ \epsilon^{\alpha\epsilon} \epsilon^{\alpha\beta} \gamma_{pq,\alpha\beta\alpha\epsilon}) = \eta T_{pq} . \qquad (3)$$

If we take $b = 2a$, these equations simplify considerably, and lead to equations which can be solved conveniently, and to which regular solutions can be chosen. So, purely for reasons of convenience, we choose $b = 2a$.

This leads to the equations

$$-\lambda\epsilon^{\alpha\beta}\gamma_{pq,\alpha\beta}-2a\lambda\epsilon^{\alpha\epsilon}\epsilon^{\alpha\beta}\gamma_{pq,\alpha\beta\alpha\epsilon}=4T_{pq}. \qquad (4)$$

Equations (4) constitute our field equations.

Let us now verify that one can choose a solution of these equations in empty space which is static, has spherical symmetry, which goes over into the flat metric at infinity, and moreover is regular at the location of the particle.

In empty space $T_{pq}=0$, and $\phi=\gamma_{44}$ is the only component of importance. Putting $2a=-1/k^2$, where $k>0$, the field equations for the static case become

$$\nabla^2\nabla^2\phi-k^2\nabla^2\phi=0 \quad .$$

In spherical coordinates,

$$\frac{d^2}{dr^2}\left(\frac{d^2}{dr^2}r\phi\right)-k^2\left(\frac{d^2}{dr^2}r\phi\right)=0.$$

The general solution to this equation is

$$\phi=\frac{A}{r}e^{-kr}+\frac{B}{r}e^{kr}+\frac{C}{r}+D.$$

For $\phi\to 0$ as $r\to\infty$, we take B = D = 0, and then

$$\phi=\frac{A}{r}e^{-kr}+\frac{C}{r}$$

If we take A = − C, then

$$\phi=\frac{C}{r}\left(e^{-kr}-1\right)$$

which is regular as $r\to 0$.

As $a\to 0$, $k\to\infty$, our equations reduce to Einstein's

equations, and ϕ approaches $-c/r$, the Schwartzschild solution, if we let $c = 2Gm$ where m is the mass of the point particle we are considering, and G is a universal constant which has the value $G = 6.6 \times 10^{-8}$ dyn cm^{2}g^{-2}.

Since we can always find a coordinate system in which the particle is at rest, ϕ will be regular at the origin in the dynamic case as well. It remains to note that the derivatives of ϕ are not analytic at the origin, for they depend on the direction of approach. It is possible to circumvent this problem by averaging over all directions, but in the final analysis, we would need to consider field equations containing fourth and sixth derivatives of the metric tensor, so as to be able to apply boundary conditions to the first derivative of ϕ.

It is possible to show that the geodesic equations of a particle follow directly from the fact that the invariant divergence of the stress-energy tensor is zero, i.e., $\sqrt{-g}\, T^{\alpha\beta}{}_{;\beta} = 0$.

The procedure is as follows.

We note that (Appendix E)

$$\frac{\partial}{\partial x^\beta} \sqrt{-g}\, T^{\alpha\beta} = -\sqrt{-g} \left\{ {\alpha \atop \mu\beta} \right\} T^{\mu\beta} \tag{5}$$

Now (Appendix F)

$$\sqrt{-g}\, T^{\alpha\beta} = m \int \frac{dz^\alpha}{ds} \frac{dz^\beta}{ds} \delta^4(x^\gamma - z^\gamma(s))\, ds$$

Therefore

$$\frac{\partial}{\partial x^\beta} (\sqrt{-g}\, T^{\alpha\beta}) = m \int \frac{dz^\alpha}{ds} \frac{dz^\beta}{ds} \frac{d}{dx^\beta} \delta^4(x^\gamma - z^\gamma(s))\, ds$$

$$= -m\int \frac{dz^\alpha}{ds}\frac{dz^\beta}{ds}\frac{d}{dz^\beta}\delta^4(x^\gamma - z^\gamma(s))\,ds$$

$$= -m\int \frac{dz^\alpha}{ds}\frac{d}{ds}\delta^4(x^\gamma - z^\gamma(s))\,ds.$$

Integrating by parts, this becomes

$$m\int \delta^4(x^\gamma - z^\gamma(s))\frac{d^2z^\alpha}{ds^2}\,ds = m\int \delta^4(x^\gamma - z^\gamma(s))\frac{d^2z^\alpha}{ds^2}\frac{ds}{dz^4}\,dz^4$$

$$= m\int \delta^3(\underline{x} - \underline{z})\,\delta(t - z^4)\frac{d^2z^\alpha}{ds^2}\frac{ds}{dz^4}\,dz^4$$

$$= m\,\delta^3(\underline{x} - \underline{z})\frac{ds}{dz^4}\frac{d^2z^\alpha}{ds^2}$$

evaluated at $z^4 = t$.

Again, $\sqrt{-g}\left\{ {\alpha \atop \mu\beta} \right\} T^{\mu\beta} = \left\{ {\alpha \atop \mu\beta} \right\} m\int \frac{dz^\mu}{ds}\frac{dz^\beta}{ds}\delta^4(x^\gamma - z^\gamma(s))\,ds$

$$= \left\{ {\alpha \atop \mu\beta} \right\} m\int \frac{dz^\mu}{ds}\frac{dz^\beta}{ds}\delta^3(\underline{x} - \underline{z})\,\delta(t - z^4)\frac{ds}{dz^4}\,dz^4$$

$$= \left\{ {\alpha \atop \mu\beta} \right\} m\,\delta^3(\underline{x} - \underline{z})\frac{ds}{dz^4}$$

evaluated at $z^4 = t$.

Substituting into equation (5) we get

$$m\frac{ds}{dz^4}\delta^3(\underline{x} - \underline{z})\left[\frac{d^2z^\alpha}{ds^2} + \left\{ {\alpha \atop \mu\beta} \right\}\frac{dz^\mu}{ds}\frac{dz^\beta}{ds}\right]_{z^4 = t} = 0.$$

If $\underline{x} \neq \underline{z}$, the delta-function is zero. If $x^\alpha = z^\alpha$, then, since ds/dz^4 is not zero, and the delta-function is infinite, we must have

$$\frac{d^2x^\alpha}{ds^2} + \left\{ {\alpha \atop \mu\beta} \right\}\frac{dx^\mu}{ds}\frac{dx^\beta}{ds} = 0 \qquad (6)$$

which is the geodesic equation for a particle of mass m.

If the Christoffel symbols are regular at the location of the particle, we can say immediately that these equations are the equations of motion of the particle. We cannot take this step in the original theory, however, for this is not true in general. But with our new field equations, we know that the Christoffel symbols are regular. Our object now is to solve these field equations to first order in λ, with the solution in integral form, and to see what happens as $a \to 0$.

We want the solutions to the equations

$$-\lambda \epsilon^{\alpha\beta} \gamma_{pq,\alpha\beta} - 2a\lambda \epsilon^{\delta\epsilon} \epsilon^{\alpha\beta} \gamma_{pq,\alpha\beta\delta\epsilon} = \eta T_{pq}.$$

Taking $a'^2 = 1/2a$ and $-\eta/2a\lambda = A$, this becomes

$$(\Box^2 + a'^2 \Box) \gamma^{\mu\nu} = A T^{\mu\nu}$$

where $\Box = \partial^2/\partial t^2 - \nabla^2$.

We are permitted to raise the indices with the $g^{\mu\nu}$ as the error in so doing is of the second order in λ. The Green's function $G(x_\rho, x_\rho')$ for this equation is defined by

$$(\Box^2 + a'^2 \Box) G(x_\rho, x_\rho') = \delta^4 (x_\rho - x_\rho').$$

Then

$$G(x_\rho, x_\rho') = \frac{1}{(2\pi)^4} \iiiint_{-\infty}^{\infty} \frac{dp_4\, dp_1\, dp_2\, dp_3\, e^{ip_\mu (x^\mu - x'^\mu)}}{p_\mu p^\mu p_\nu p^\nu - a'^2 p_\mu p^\mu}$$

as can be seen by direct substitution. This can be factored to give

$$G(x_\rho, x_\rho') = \frac{1}{(2\pi)^4 a'^2} \left[\iiiint_{-\infty}^{\infty} \frac{d^4 p\, e^{ip_\mu (x^\mu - x'^\mu)}}{p_4^2 - p^2 - a^2} \right.$$

$$- \iiiint_{-\infty}^{\infty} \frac{d^4p \, e^{ip_\mu (x^\mu - x'^\mu)}}{p_4^{\,2} - p^2} \Big]$$

H.J. Bhabha (5) solves a similar equation, and gets for his retarded Green's function

$$- \frac{1}{(2\pi)^4} \iiiint_{-\infty}^{\infty} \frac{d^4p \, e^{ip_\mu (x^\mu - x'^\mu)}}{p_4^{\,2} - p^2 - \chi^2}$$

$$= \begin{pmatrix} 2\delta(u^2) - \frac{\chi}{u} J_1(\chi u) \\ 0 \\ 0 \end{pmatrix} \begin{matrix} u_4 > u_r \\ |u_4| < u_r \\ u_4 < -u_r \end{matrix}$$

where $u_\rho = x_\rho - x'_\rho$, $u_r = \sum_{k=1}^{3} (-u_k u^k)^{1/2}$

and $u^2 = u_4^2 - u_r^2$.

Substituting, we get for our retarded Green's function $G(x_\rho, x'_\rho)$

$$= -\frac{1}{a'^2} \left[\begin{pmatrix} 2\delta(u^2) - \frac{a'}{u} J_1(a'u) \\ 0 \\ 0 \end{pmatrix} - \begin{pmatrix} 2\delta(u^2) \\ 0 \\ 0 \end{pmatrix} \right] \begin{matrix} u_4 > u_r \\ |u_4| < u_r \\ u_4 < -u_r \end{matrix}$$

$$= \begin{pmatrix} \frac{1}{a'u} J_1(a'u) \\ 0 \\ 0 \end{pmatrix} \begin{matrix} u_4 > u_r \\ |u_4| < u_r \\ u_4 < -u_r \end{matrix} .$$

Now

$$\gamma^{\mu\nu}(x_\rho) = A\int_{-\infty}^{\infty} dx'_\rho\, G(x_\rho, x'_\rho)\, T^{\mu\nu}(x'_\rho)$$

and

$$\sqrt{-g}\, T^{\mu\nu}(x'_\rho) = m\int \dot{z}^\mu \dot{z}^\nu \delta^4(x'_\rho - z_\rho)\, ds$$

Therefore

$$\gamma^{\mu\nu}(x_\rho) = Am\int_{-\infty}^{\infty} dx'_\rho \frac{1}{au}\frac{J_1(au)}{\sqrt{-g}}\int_{-\infty}^{\infty} ds\, \dot{z}^\mu \dot{z}^\nu \delta^4(x'_\alpha - z_\alpha)$$

$$= Am\int_{-\infty}^{\infty} ds \frac{1}{au(s)}\frac{J_1(au(s))}{\sqrt{-g}}\dot{z}^\mu(s)\dot{z}^\nu(s)$$

where in the last expression $u_\rho = x_\rho - z_\rho$.

We now, for convenience, introduce another constant, b , defined by $b^2 = 2a = 1/a'^2$. Then $A = -\eta/2a\lambda = -\eta/b^2\lambda$

and

$$\gamma^{\mu\nu}(x_\rho) = -\frac{\eta}{\lambda}\left(\frac{m}{b}\right)\int_{-\infty}^{\infty} ds \frac{1}{u(s)}\frac{J_1\left(\frac{u(s)}{b}\right)}{\sqrt{-g}}\dot{z}^\mu(s)\dot{z}^\nu(s).$$

This is the retarded solution in integral form that we were looking for. We know that as a→ 0, equation (4) approaches that of Einstein if we take $\eta = 16\pi G$ (Appendix G). Let us now see if we arrive at his solution if we let a approach zero in our solution. In this case, $b \to 0$ as well, and in order to have a finite answer, we must let m approach zero with b such that m/b is a constant, m'. Also, as b→0, $J_1(u(s)/b)/u(s)$ becomes a Dirac delta-function and the integral can be solved exactly to give

$$\gamma^{\mu\nu}(x_\rho) = -\frac{\eta}{\lambda}\frac{m'\dot{z}^\mu(s_0)\dot{z}^\nu(s_0)}{k\sqrt{-g}\,\big|_{s_0}} \qquad \text{(Appendix H).}$$

where s_0 is the proper time of the retarded point, i.e. the point on the world line such that $u_\mu u^\mu = 0$ and $u_4 > 0$. In the future we shall make the convention that if a function of s is not under an integral with respect to s , then it is taken at $s = s_0$. k is defined by

$$K = -u_\alpha du^\alpha/ds = (x_\alpha - z_\alpha)\dot{z}^\alpha = u_\alpha v^\alpha$$

where a dot above a function of s denotes differentiation with respect to s, and $v^\alpha = \dot{z}^\alpha$.

To first order in λ, $\sqrt{-g} = 1 + (\lambda/2)\epsilon^{\rho\sigma}h_{\rho\sigma}$, and so to this order, $\gamma^{\mu\nu}(x_\rho) = -(4/\lambda)(m'/K)v^\mu v^\nu$. (7)

To fix m', look at the 44 component of $\gamma^{\mu\nu}(x_\rho)$ in the static case, and compare with the expression given in the original theory, i.e.

$$\lambda\gamma_{44}(\bar{r}) = -4G\int_{V'}\frac{\rho(\bar{r}')dV'}{|\bar{r}-\bar{r}'|} = -4G\int_{V'}\frac{m\delta(\bar{r}'-z)dV'}{|\bar{r}-\bar{r}'|} = \frac{-4Gm}{|\bar{r}-\bar{z}|}.$$

In this case, $v^\alpha = (0,0,0,1)$, and $K = u_4 = u^4$. Also, since $u_\mu u^\mu = 0$, $u_4^2 = \sum_i u_i^2$, (i 1,2,3). We also note that $u_4 = x_4 - z_4(s_0)$ is positive, since we are considering values at the proper time of the retarded point. Then we have

$$\lambda\gamma^{44} = -\frac{4m'v^4v^{-4}}{u_4 v^4} = \frac{-16\pi Gm'}{x_4 - z_4} = -16\pi Gm'/|\bar{r}-\bar{z}|.$$

Comparing these two expressions, it follows that $m' = m/4\pi$.

We will now exhibit the Christoffel symbols derived from our expression for $\gamma^{\mu\nu}$. Letting $B = 2Gm$, we get (Appendix I)

$$\begin{aligned}\left\{{\rho \atop \mu\nu}\right\} = B\Big\{&\frac{1}{K^2}\big[u_\mu v_\nu \dot{v}^\rho + u_\mu \dot{v}_\nu v^\rho + u_\nu \dot{v}_\mu v^\rho + u_\nu v_\mu \dot{v}^\rho \\ &- \dot{v}_\mu v_\nu u^\rho - \dot{v}_\nu v_\mu u^\rho - v_\nu v_\mu v^\rho\big] \\ &+ \frac{1-K'}{K^3}\big[v_\nu v^\rho u_\mu + v_\mu v^\rho u_\nu - v_\mu v_\nu u^\rho\big] \\ &+ \frac{1}{2}\frac{(1-K')}{K^3}\big[\epsilon_{\mu\nu}u^\rho - \delta_\mu^\rho u_\nu - \delta_\nu^\rho u_\mu\big] \\ &- \frac{1}{2}\cdot\frac{1}{K^2}\big[\epsilon_{\mu\nu}v^\rho - \delta_\mu^\rho v_\nu - \delta_\nu^\rho v_\mu\big]\Big\}\end{aligned} \quad (8)$$

where $K' = u_\mu v^\mu$.

As a check on this formula, let us again consider the case of a stationary particle. Equation (8) reduces to

$$\left\{ {\rho \atop \mu\nu} \right\} = \frac{B}{(u_4)^2}\left[\frac{1}{u_4}\left(v^\rho v_\nu u_\mu + v_\mu v^\rho u_\nu - v_\mu v_\nu u^\rho\right) + \frac{1}{2u_4}\left(\epsilon_{\mu\nu} u^\rho - u_\nu \delta_\mu^\rho - u_\mu \delta_\nu^\rho\right) - \frac{1}{2}\left(\epsilon_{\mu\nu} v^\rho - v_\nu \delta_\mu^\rho - v_\mu \delta_\nu^\rho\right) - v^\rho v_\nu v_\mu\right].$$

This gives, where $\rho \neq \mu \neq \nu \neq 4$,

$$\left\{ {\rho \atop \mu\mu} \right\} = \left\{ {\rho \atop 44} \right\} = -\left\{ {\rho \atop \rho\rho} \right\} = -\left\{ {\nu \atop \nu\rho} \right\} = \left\{ {4 \atop 4\rho} \right\} = -\frac{B}{2}\frac{u^\rho}{(u_4)^3}$$

and

$$\left\{ {4 \atop 44} \right\} = \left\{ {4 \atop \mu\mu} \right\} = \left\{ {\rho \atop 4\rho} \right\} = \left\{ {\rho \atop 4\nu} \right\} = \left\{ {\rho \atop \mu\nu} \right\} = \left\{ {4 \atop \mu\nu} \right\} = 0.$$

Also, under the above conditions, it is easy to show that

$$\gamma_{44,\rho} = -\frac{2B}{\lambda}\frac{u^\rho}{(u_4)^3}.$$

So $-\frac{B u^\rho}{2(u_4)^3} = \frac{\lambda}{4}\gamma_{44,\rho}$.

In our theory, λ served as an expansion parameter. Nevertheless, we may now choose a definite value for it, and we take it equal to unity. Then the Christoffel symbols check with the values calculated by Einstein, Infeld, and Hoffman (see, e.g., Bergmann p.230).

It is now possible to consider the case of two interacting point particles. Let z_1^α, z_2^α label the coordinates of particles 1,2 respectively, and let x^α label the coordinates of the field point.

Then, if s_1, s_2 be the world-line parameters of the two particles, we have, writing down the geodesic equations of each of them,

$$\frac{d^2 z_1^\alpha}{ds_1^2} + \left\{ {\alpha \atop \mu\beta} \right\}(x^\alpha, z_1^\alpha) \frac{dz_1^\mu}{ds_1} \frac{dz_1^\beta}{ds_1} = 0$$

$$\frac{d^2 z_2^\alpha}{ds_2^2} + \left\{ {\alpha \atop \mu\beta} \right\}(x^\alpha, z_2^\alpha) \frac{dz_2^\mu}{ds_2} \frac{dz_2^\beta}{ds_2} = 0 .$$

Now let $x^\alpha \to z_2^\alpha$ in the first equation, and let $x^\alpha \to z_1^\alpha$ in the second. These then represent the equations of motion of the two particles.

SUMMARY

It was shown that the invariant divergence vanishes identically for the tensor density that is constituted by the Hamiltonian derivatives of any scalar density that depends only on the metric tensor and its derivatives with respect to the coordinates up to any finite order. This tensor density could then be identified with the product of the stress-energy tensor with the square root of the determinant of the metric tensor, this identification to be considered as a possible set of field equations was derived which reduces to Einstein's equations as a certain parameter "a" tends to zero.

The linear approximation was applied, and a solution was found for the static case which was regular at the location of the particle. This meant that the Christoffel symbols were also regular there. It was then shown that the geodesic equations of a particle follow directly from the vanishing of the invariant divergence of the stress-energy tensor, or, we can say, directly from the field equations themselves. These equations contain the Christoffel symbols, which were shown to be regular at the location of the particle; so the geodesic equations can now be called equations of motion. As the parameter "a" tends to zero, the Christoffel symbols approach singularity in a regular way; this could be called a regulator technique.

In the latter part of the paper, the time-dependent field equations were solved in the linear approximation with the solution

in integral form. Then, letting "a" approach zero, the integration was carried out, and the solution was shown to approach that of Einstein for the static case. Also, the Christoffel symbols were exhibited explicitly, and the case of a stationary particle was given special consideration.

In conclusion, the new field equations lead to a Riemannian Geometry satisfying the principle of equivalence in which the equations of motion follow from the field equations and are geodesics.

APPENDIX A

To show that if

$$I = \int \mathfrak{M}\, dx^4$$

where $\mathfrak{M}$ is any scalar density that depends only on the g_{ik} and their derivatives up to any finite order, and we compute δI and perform the partial integrations needed to put it into the form

$$\delta I = \int \frac{\delta \mathfrak{M}}{\delta g_{ik}} \delta g_{ik}\, dx^4, \tag{A.1}$$

then $\mathfrak{M}^{ik}{}_{;i} \equiv 0$ where $\mathfrak{M}^{ik} = \delta \mathfrak{M} / \delta g_{ik}$.

Since δI is the difference between two invariants, it itself is an invariant. Then $\mathfrak{M}^{ik} \delta g_{ik}$ is a scalar density. But δg_{ik} , being the difference of two symmetric second rank tensors, is itself a symmetric second rank tensor. So $\mathfrak{M}^{ik}$ is a symmetric, second rank, contravariant tensor density.

Now we bring about a variation in g_{ik} by changing the frame, i.e. we let $x_i = x'_\ell + \lambda \phi_\ell (x'_i) + \lambda^2 \psi_\ell (x'_i)$ where λ is an expansion parameter. This expansion approaches identity as $\lambda \to 0$. The integral then becomes

$$I = \int \mathfrak{M} (g'_{\ell m}(x'_r), \ldots)\, dx'^4$$

where $\mathfrak{M}$ is the same function of the $g'_{\ell m}(x'_r) = \frac{\partial x_\ell}{\partial x'_i} \frac{\partial x_m}{\partial x'_k} g_{\ell m}(x_s)$ and their derivatives with respect to the x' as it was before of

the $g_{lm}(x_n)$ and their derivatives with respect to the x. The limits of integration are the same if we ask that the transformation shall approach to identity at the boundary for any λ. The only formal change is that the argument is now not $g_{ik}(x'_s)$ but $g'_{ik}(x'_s)$.

Expanding with respect to λ,

$$g'_{ik}(x'_s) - g_{ik}(x_s) = \lambda\left[g_{lk}\frac{\partial\phi_l}{\partial x'_i} + g_{im}\frac{\partial\phi_m}{\partial x'_k} + \frac{\partial g_{ik}}{\partial x'_n}\phi_n\right] + O(\lambda^2),$$

all functions written with the x'. Since this must vanish at the boundary for any λ, use equation (A.1) with first order terms in λ for δg_{ik}. Dropping superfluous dashes we get

$$\delta I = \int \mathfrak{m}^{ik}\left(g_{lk}\frac{\partial\phi_l}{\partial x_i} + g_{im}\frac{\partial\phi_m}{\partial x_k} + \frac{\partial g_{ik}}{\partial x_n}\phi_n\right)dx^4 \equiv 0.$$

Integrating by parts,

$$0 \equiv \int\left(-\frac{\partial \mathfrak{m}^i{}_l}{\partial x_i}\phi_l - \frac{\partial \mathfrak{m}^k{}_m}{\partial x_k}\phi_m + \mathfrak{m}^{ik}\frac{\partial g_{ik}}{\partial x_n}\phi_n\right)dx^4$$

$$\equiv \int\left(-2\frac{\partial \mathfrak{m}^k{}_m}{\partial x_k} + \mathfrak{m}^{ik}\frac{\partial g_{ik}}{\partial x_m}\right)\phi_m\,dx^4.$$

Now ϕ_m is arbitrary, hence

$$\frac{\partial \mathfrak{m}^k{}_m}{\partial x_k} - \frac{1}{2}\mathfrak{m}^{ik}\frac{\partial g_{ik}}{\partial x_m} \equiv 0.$$

Using the symmetry of $\mathfrak{m}^{ik}$,

$$\frac{1}{2}\mathfrak{m}^{ik}\frac{\partial g_{ik}}{\partial x_m} = \frac{1}{2}\mathfrak{m}^{ik}\left(\frac{\partial g_{im}}{\partial x_k} + \frac{\partial g_{ik}}{\partial x_m} - \frac{\partial g_{km}}{\partial x_i}\right)$$

$$= \mathfrak{m}^k_i\left\{{i \atop k\,m}\right\}.$$

where $\left\{ {i \atop km} \right\}$ is the Christoffel symbol of the second kind. Thus we get

$$\frac{\partial \mathfrak{M}^k{}_m}{\partial x_k} - \mathfrak{M}^k{}_i \left\{ {i \atop k\,m} \right\} = \mathfrak{M}^k{}_{m;k} \equiv 0.$$

We may also write, since $g^{lm}{}_{;k} = 0$,

$$\mathfrak{M}^{km}{}_{;k} \equiv 0.$$

APPENDIX B

To show that if we take $\mathfrak{M} = F(g^{ik}, R_{ik})\sqrt{-g}$ in our Hamiltonian principle, we arrive at the identity

$$\left[(g^{\alpha\epsilon}\beta^{ik} + g^{ki}\beta^{\epsilon\alpha} - g^{\alpha i}\beta^{\epsilon k} - g^{\alpha k}\beta^{i\epsilon})_{;\alpha\epsilon} + Fg^{ik} + 2\frac{\delta F}{\delta g_{ik}}\right]_{;i} \equiv 0$$

where $\beta^{ik} = \dfrac{\delta F}{\delta R_{ik}}$.

Consider $I = \int F\sqrt{-g}\,dx^4$. We compute δI .

$$\delta I = \int\left(\frac{\delta F}{\delta R_{ik}}\delta R_{ik}\right)\sqrt{-g}\,dx^4 + \int\left(\frac{\delta F}{\delta g_{ik}}\delta g_{ik}\right)\sqrt{-g}\,dx^4$$

$$+ \int F\,\delta(\sqrt{-g})\,dx^4.$$

We shall call these the first, second, and third integrals respectively. Use is made of the following relations:

$$\frac{\delta g}{g} = \frac{\delta(-g)}{(-g)} = \frac{2\delta(\sqrt{-g})}{\sqrt{-g}} = g^{ik}\delta g_{ik}$$ (Schrödinger (4), p.97)

$$\delta R_{ik} = -(\delta\Gamma^{\alpha}_{ik})_{;\alpha} + (\delta\Gamma^{\alpha}_{i\alpha})_{;k}$$ (Schrödinger (4), p.98)

where $\Gamma^{\alpha}_{\beta\gamma} \equiv \left\{\begin{matrix}\alpha\\ \beta\gamma\end{matrix}\right\}$.

$$g_{;\alpha} = 0$$

$$\delta\Gamma^{\alpha}_{ik} = \delta\left\{\tfrac{1}{2}g^{\alpha l}\left[g_{kl,i} + g_{li,k} - g_{ik,l}\right]\right\}$$

$$= [ik,l]\,\delta g^{\alpha l} + \tfrac{1}{2}g^{\alpha l}(\delta g_{kl,i} + \delta g_{li,k} - \delta g_{ik,l}).$$

The first integral then becomes

$$\int \frac{\delta F}{\delta R_{ik}} \sqrt{-g}\left[(\delta T^{\alpha}_{i\alpha})_{;k} - (\delta T^{\alpha}_{ik})_{;\alpha} \right] dx^4$$

which is, upon integrating by parts with respect to the semi-colons (Schrodinger (4), p.36), and noting that $g_{;\alpha} = 0$,

$$\int \left\{ \sqrt{-g}\left(\frac{\delta F}{\delta R_{ik}}\right)_{;\alpha} \delta T^{\alpha}_{ik} - \sqrt{-g}\left(\frac{\delta F}{\delta R_{il}}\right)_{;l} \delta^{k}_{\alpha} \delta T^{\alpha}_{ik} \right\} dx^4$$

$$= \int \mathcal{H}^{ik}_{\alpha}\, \delta T^{\alpha}_{ik}\, dx^4 \qquad \text{(B.1)}$$

where

$$\mathcal{H}^{ik}_{\alpha} = \sqrt{-g}\left[\left(\frac{\delta F}{\delta R_{ik}}\right)_{;\alpha} - \left(\frac{\delta F}{\delta R_{il}}\right)_{;l} \delta^{k}_{\alpha} \right]$$

is a tensor density.

Now, using the relation $\delta g^{\alpha l} = -g^{r\alpha} g^{sl} \delta g_{rs}$, we have

$$\mathcal{H}^{ik}_{\alpha}[ik,l]\,\delta g^{\alpha l} = -\mathcal{H}^{ik}_{\alpha}[ik,l]\, g^{r\alpha} g^{sl} \delta g_{rs} = -\mathcal{H}^{ikr}\left\{ {s \atop ik} \right\} \delta g_{rs}$$

where we have written $\mathcal{H}^{ikr} = g^{\alpha r} \mathcal{H}^{ik}_{\alpha}$.

Using our relation for δT^{α}_{ik}, substituting into (B.1), and integrating by parts, we get

$$-\int \Big[\mathcal{H}^{rsi}\left\{ {k \atop rs} \right\} \delta g_{ik} + \tfrac{1}{2}\mathcal{H}^{ikl}{}_{,i}\, \delta g_{kl} + \tfrac{1}{2}\mathcal{H}^{ikl}{}_{,k}\, \delta g_{li}$$
$$- \tfrac{1}{2}\mathcal{H}^{ikl}{}_{,l}\, \delta g_{ik} \Big] dx^4$$

$$= -\int \Big[\mathcal{H}^{\epsilon li}\left\{ {k \atop \epsilon l} \right\} + \tfrac{1}{2}\mathcal{H}^{\epsilon ki}{}_{,\epsilon} + \tfrac{1}{2}\mathcal{H}^{i\epsilon k}{}_{,\epsilon} - \tfrac{1}{2}\mathcal{H}^{ik\epsilon}{}_{,\epsilon} \Big] \delta g_{ik}\, dx^4.$$

Now

$$\mathcal{H}^{\epsilon k i}{}_{;\epsilon} = \mathcal{H}^{\epsilon k i}{}_{,\epsilon} + \left\{ {\epsilon \atop \epsilon l} \right\} \mathcal{H}^{l k i} + \left\{ {k \atop \epsilon l} \right\} \mathcal{H}^{\epsilon l i}$$
$$+ \left\{ {i \atop \epsilon l} \right\} \mathcal{H}^{\epsilon k l} - \left\{ {l \atop l \epsilon} \right\} \mathcal{H}^{\epsilon k i},$$

following the rule for covariant differentiation of a tensor density (Schrodinger (4), p.33). Using the symmetry in ϵ and l in $\left\{ {k \atop \epsilon l} \right\}$, and the symmetry in i and k in each term, substitution yields

$$-\frac{1}{2}\int (\mathcal{H}^{\epsilon k i} + \mathcal{H}^{i \epsilon k} - \mathcal{H}^{i k \epsilon})_{;\epsilon}\, \delta g_{ik}\, dx^4.$$

The first integral is now in covariant form.

Using $\frac{\delta(\sqrt{-g})}{\sqrt{-g}} = \frac{1}{2} g^{ik} \delta g_{ik}$, the third integral becomes

$$\int \frac{1}{2} F \sqrt{-g}\, g^{ik}\, \delta g_{ik}\, dx^4.$$

Combining these results,

$$0 = \delta I = \int \Big\{ -\frac{1}{2}(\mathcal{H}^{\epsilon k i} + \mathcal{H}^{i \epsilon k} - \mathcal{H}^{i k \epsilon})_{;\epsilon}$$
$$+ \frac{1}{2} F \sqrt{-g}\, g^{ik} + \frac{\delta F}{\delta g_{ik}} \sqrt{-g} \Big\}\, \delta g_{ik}\, dx^4. \quad \text{(B.2)}$$

The expression in parenthesis corresponds to our $\mathcal{M}^{ik}$. So, since $\mathcal{M}^{ik}{}_{;i} \equiv 0$, we divide by $\sqrt{-g}$ under the semi-colon and multiply by 2 to get

$$\left[-\frac{1}{\sqrt{-g}}(\mathcal{H}^{\epsilon k i} + \mathcal{H}^{i \epsilon k} - \mathcal{H}^{i k \epsilon})_{;\epsilon} + F g^{ik} + \frac{2\delta F}{\delta g_{ik}} \right]_{;i} \equiv 0. \quad \text{(B.3)}$$

Remembering our definitions of β^{ik}, $\mathcal{H}^{ik}_{\alpha}$, and $\mathcal{H}^{ik\epsilon}$, we obtain $\mathcal{H}^{ik\epsilon} = \sqrt{-g}\,(g^{\alpha\epsilon} \beta^{ik}{}_{;\alpha} - g^{k\epsilon} \beta^{il}{}_{;l})$.

So equation (B.3) becomes

$$\Big[(g^{\alpha\epsilon}\beta^{ik} + g^{ki}\beta^{\epsilon\alpha} - g^{\alpha i}\beta^{\epsilon k} - g^{\alpha k}\beta^{i\epsilon})_{;\alpha\epsilon}$$
$$+ F g^{ik} + 2\frac{\delta F}{\delta g_{ik}} \Big]_{;i} \equiv 0.$$

APPENDIX C

To show that taking $F = R + aR^2 + bR_{ij}R^{ij}$ leads to the equation

$$Q_{pq}(R) + a\,Q_{pq}(R^2) + b\,Q_{pq}(R_{ij}R^{ij}) = g_{ip}g_{kq}\eta T^{ik} = \eta T_{pq}$$

where

$$Q_{pq}(R) = -2\left(R_{pq} - \tfrac{1}{2}g_{pq}R\right)$$

$$Q_{pq}(R^2) = 2\left(g^{ik}g^{\alpha\epsilon} + g^{\epsilon\alpha}g^{ki} - g^{\epsilon k}g^{\alpha i} - g^{i\epsilon}g^{\alpha k}\right)R_{;\alpha\epsilon} + R^2 g^{ik} - 4RR^{ik}$$

$$Q_{pq}(R_{ij}R^{ij}) = 2\left(g^{\alpha\epsilon}\delta^{\mu}_{p}\delta^{\nu}_{q} + g_{pq}g^{\mu\epsilon}g^{\nu\alpha} - \delta^{\alpha}_{p}\delta^{\nu}_{q}g^{\epsilon\mu} - g^{\nu\epsilon}\delta^{\alpha}_{q}\delta^{\mu}_{p}\right)R_{\mu\nu;\alpha\epsilon} + R_{\mu\nu}R^{\mu\nu}g_{pq} - 2R^{\alpha k}R_{\alpha p}g_{kq}.$$

For brevity, introduce the following notation:

$$P^{ik}(A) = \left(g^{\alpha\epsilon}\beta^{ik} + g^{ki}\beta^{\epsilon\alpha} - g^{\alpha i}\beta^{\epsilon k} - g^{\alpha k}\beta^{i\epsilon}\right)_{;\alpha\epsilon}$$

$$P^{ik}(B) = Fg^{ik}$$

$$P^{ik}(C) = 2\,\delta F/\delta g_{ik}$$

$P^{ik}(A,R) = P^{ik}(A)$ **with** $F = R$; $P^{ik}(B,R^2) = P^{ik}(B)$ **with** $F = R^2$ **etc.**

$$Q_{pq}(R) = g_{ip}g_{kq}\left(P^{ik}(A,R) + P^{ik}(B,R) + P^{ik}(C,R)\right)$$

$$Q_{pq}(R^2) = g_{ip}g_{kq}\left(P^{ik}(A,R^2) + P^{ik}(B,R^2) + P^{ik}(C,R^2)\right)$$

etc.

Then, using the expressions for β^{ik} and $\delta F/\delta g_{ik}$ listed in the

text, we get $P^{ik}(A,R)=0$ since $g^{ik}{}_{;\alpha}=0$, and

$$P^{ik}(B,R)+P^{ik}(C,R)=Rg^{ik}-2R_{\alpha\beta}g^{\alpha i}g^{\beta k}.$$

Therefore

$$Q_{pq}(R)=-2\left(R_{pq}-\tfrac{1}{2}g_{pq}R\right).$$

Again,

$$\begin{aligned} &P^{ik}(A,R^2)+P^{ik}(B,R^2)+P^{ik}(C,R^2)\\ &=2\left(g^{ik}g^{\alpha\epsilon}+g^{\epsilon\alpha}g^{ki}-g^{\epsilon k}g^{\alpha i}-g^{i\epsilon}g^{\alpha k}\right)R_{;\alpha\epsilon}\\ &\quad+R^2g^{ik}-4RR^{ik}\\ &=4\left(g^{ik}g^{\alpha\epsilon}-g^{\epsilon k}g^{\alpha i}\right)R_{;\alpha\epsilon}+R^2g^{ik}-4RR_{\alpha\beta}g^{i\alpha}g^{k\beta}. \end{aligned}$$

Therefore

$$Q_{pq}(R^2)=4\left(g_{pq}g^{\alpha\epsilon}-\delta_q^{\epsilon}\delta_p^{\alpha}\right)R_{;\alpha\epsilon}+R^2g_{pq}-4RR_{pq}.$$

Again,

$$\begin{aligned} &P^{ik}(A,R_{ij}R^{ij})+P^{ik}(B,R_{ij}R^{ij})+P^{ik}(C,R_{ij}R^{ij})\\ &=2\left(g^{\alpha\epsilon}R^{ik}+g^{ki}R^{\epsilon\alpha}-g^{\alpha i}R^{\epsilon k}-g^{\alpha k}R^{i\epsilon}\right)_{;\alpha\epsilon}\\ &\quad+R_{\mu\nu}R^{\mu\nu}g^{ik}-2R^{\alpha k}R_{\alpha\beta}g^{i\beta}\\ &=2\left(g^{\alpha\epsilon}g^{\mu i}g^{\nu k}R_{\mu\nu}+g^{ki}g^{\mu\epsilon}g^{\nu\alpha}R_{\mu\nu}-g^{\alpha i}g^{\epsilon\mu}g^{k\nu}R_{\mu\nu}\right.\\ &\quad\left.-g^{\alpha k}g^{\mu i}g^{\nu\epsilon}R_{\mu\nu}\right)_{;\alpha\epsilon}+R_{\mu\nu}R^{\mu\nu}g^{ik}-2R^{\alpha k}R_{\alpha\beta}g^{i\beta}\\ &=2\left(g^{\alpha\epsilon}g^{\mu i}g^{\nu k}+g^{ki}g^{\mu\epsilon}g^{\nu\alpha}-g^{\alpha i}g^{\epsilon\mu}g^{k\nu}\right.\\ &\quad\left.-g^{\alpha k}g^{\mu i}g^{\nu\epsilon}\right)R_{\mu\nu;\alpha\epsilon}+R_{\mu\nu}R^{\mu\nu}g^{ik}-2R^{\alpha k}R_{\alpha\beta}g^{i\beta}. \end{aligned}$$

Therefore

$$Q_{pq}(R_{ij}R^{ij}) = 2\,(g^{\alpha\epsilon}\delta_p^{\mu}\delta_q^{\nu} + g_{pq}g^{\mu\epsilon}g^{\nu\alpha} - \delta_p^{\alpha}\delta_q^{\nu}g^{\epsilon\mu} - g^{\nu\epsilon}\delta_q^{\alpha}\delta_p^{\mu})R_{\mu\nu;\alpha\epsilon} + R_{\mu\nu}R^{\mu\nu}g_{pq} - 2R^{\alpha k}R_{\alpha p}g_{kq}.$$

Substituting into equation (1), we have

$$Q_{pq}(R) + a\,Q_{pq}(R^2) + b\,Q_{pq}(R_{ij}R^{ij}) = \eta T_{pq}$$

where the Q_{pq}'s are as at the top of page 24 .

APPENDIX D

To show that in the linear approximation, in a system in which

$T_\mu = 0$,

$Q_{pq}(R)$ reduces to $-\lambda \epsilon^{a\rho} \gamma_{;pq,a\rho}$

$Q_{pq}(R^2)$ reduces to $-2\lambda \epsilon_{pq} \epsilon^{\alpha\epsilon} \epsilon^{a\rho} \gamma_{;,a\rho\alpha\epsilon} + 2\lambda \epsilon^{a\rho} \gamma_{;,a\rho pq}$

and $Q_{pq}(R_{ij}R^{ij})$ reduces to $\lambda [\epsilon^{\alpha\epsilon} \epsilon^{a\rho} \gamma_{;pq,a\rho\alpha\epsilon} - \epsilon_{pq} \epsilon^{a\rho} \epsilon^{\alpha\epsilon} \gamma_{;,a\rho\alpha\epsilon}$

$+ \epsilon^{a\rho} \gamma_{;,a\rho pq}]$

where all quantities are as defined in the text.

To first order in λ,

$$\left\{ {\rho \atop \mu\nu} \right\} = \frac{\lambda}{2} \epsilon^{a\rho} (h_{;a\nu,\mu} + h_{;\mu a,\nu} - h_{;\mu\nu,a})$$

$$\left\{ {\rho \atop \mu\nu} \right\}_{,\rho} = \frac{\lambda}{2} \epsilon^{a\rho} (h_{;a\nu,\mu\rho} + h_{;\mu a,\nu\rho} - h_{;\mu\nu,a\rho})$$

$$\left\{ {\rho \atop \nu\rho} \right\} = \frac{\lambda}{2} \epsilon^{a\rho} h_{;a\rho,\nu}$$

$$\left\{ {\rho \atop \nu\rho} \right\}_{,\mu} = \frac{\lambda}{2} \epsilon^{a\rho} h_{;a\rho,\nu\mu} = \frac{\lambda}{2} h_{;,\nu\mu}.$$

So, to the same order,

$$\{\ \}\{\ \} = \{\ \}_{,\mu}\{\ \} = \{\ \}_{,\mu}\{\ \}_{,\nu} = 0.$$

$$R_{\mu\nu} = \left\{ {\rho \atop \nu\rho} \right\}_{,\mu} - \left\{ {\rho \atop \mu\nu} \right\}_{,\rho} - \left\{ {\sigma \atop \rho\sigma} \right\}\left\{ {\rho \atop \mu\nu} \right\} + \left\{ {\sigma \atop \mu\rho} \right\}\left\{ {\rho \atop \nu\sigma} \right\}$$

$= \left\{ {\rho \atop \nu\rho} \right\}_{,\mu} - \left\{ {\rho \atop \mu\nu} \right\}_{,\rho}$ to first order in λ.

$R_{\mu\nu;\alpha} = R_{\mu\nu,\alpha} - \left\{ {a \atop \mu\alpha} \right\} R_{a\nu} - \left\{ {a \atop \nu\alpha} \right\} R_{\mu a} = R_{\mu\nu,a}$ to first order.

Similarly, $R_{\mu\nu;\alpha\epsilon} = R_{\mu\nu,\alpha\epsilon}$ at most, to first order

$$= \left\{ {\rho \atop \nu\rho} \right\}_{,\mu\alpha\epsilon} - \left\{ {\rho \atop \mu\nu} \right\}_{,\rho\alpha\epsilon} \quad \text{to first order}$$

$$= \frac{\lambda}{2}\Big[h_{,\nu\mu\alpha\epsilon} - \epsilon^{a\rho}\big(h_{a\nu,\mu\rho\alpha\epsilon} + h_{\mu a,\nu\rho\alpha\epsilon} - h_{\mu\nu,a\rho\alpha\epsilon}\Big].$$

Now, substituting for $h_{\mu\nu}$ using $h_{\mu\nu} = \gamma_{\mu\nu} - \frac{1}{2}\epsilon_{\mu\nu}\gamma$ and $\gamma = -h = \epsilon^{\mu\nu}\gamma_{\mu\nu}$,

$$R_{\mu\nu;\alpha\epsilon} = \frac{\lambda}{2}\Big[-\gamma_{,\nu\mu\alpha\epsilon} - \epsilon^{a\rho}\big(\gamma_{a\nu,\mu\rho\alpha\epsilon} - \tfrac{1}{2}\epsilon_{a\nu}\gamma_{,\mu\rho\alpha\epsilon} + \gamma_{\mu a,\nu\rho\alpha\epsilon} - \tfrac{1}{2}\epsilon_{\mu a}\gamma_{,\nu\rho\alpha\epsilon} - \gamma_{\mu\nu,a\rho\alpha\epsilon} + \tfrac{1}{2}\epsilon_{\mu\nu}\gamma_{,a\rho\alpha\epsilon}\big)\Big]$$

$$= \frac{\lambda}{2}\Big[\cancel{-\gamma_{,\nu\mu\alpha\epsilon}} - \epsilon^{a\rho}\gamma_{\nu a,\rho\mu\alpha\epsilon} + \cancel{\tfrac{1}{2}\delta_\nu^\rho\gamma_{,\mu\rho\alpha\epsilon}} - \epsilon^{a\rho}\gamma_{\mu a,\rho\nu\alpha\epsilon} + \cancel{\tfrac{1}{2}\delta_\mu^\rho\gamma_{,\nu\rho\alpha\epsilon}} + \epsilon^{a\rho}\gamma_{\mu\nu,a\rho\alpha\epsilon} - \tfrac{1}{2}\epsilon^{a\rho}\epsilon_{\mu\nu}\gamma_{,a\rho\alpha\epsilon}\Big]$$

$$= \frac{\lambda}{2}\Big[-\sigma_{\nu,\mu\alpha\epsilon} - \sigma_{\mu,\nu\alpha\epsilon} + \epsilon^{a\rho}\gamma_{\mu\nu,a\rho\alpha\epsilon} - \tfrac{1}{2}\epsilon_{\mu\nu}\gamma_{,a\rho\alpha\epsilon}\Big].$$

We can find (see Bergmann (6), p.183) a coordinate system in which $\sigma_\mu = \epsilon^{a\rho}\gamma_{\mu a,\rho} = 0.$

So $R_{\mu\nu;\alpha\epsilon} = \frac{\lambda}{2}[\epsilon^{a\rho}\gamma_{\mu\nu,a\rho\alpha\epsilon} - \frac{1}{2}\epsilon_{\mu\nu}\gamma_{,a\rho\alpha\epsilon}]$ to first order in a system in which $\sigma_\mu = 0$.

We have incidentally shown that to this order, R^2, $R_{\mu\nu}R^{\mu\nu}$, $R_{ab}R^{cd}$, and $R_{ij}R_{ij}$ all vanish.

So, in the linear approximation,

$$Q_{pg}(R) = -2\cdot\frac{\lambda}{2}\epsilon^{a\rho}\gamma_{pg,a\rho}$$ (see Bergmann (6) pp. 182, 183)

$$Q_{pg}(R^2) = 4(\epsilon_{pg}\epsilon^{\alpha\epsilon} - \delta_g^\epsilon\delta_p^\alpha)\epsilon^{\mu\nu}R_{\mu\nu;\alpha\epsilon}$$

$$= 4\cdot\frac{\lambda}{2}(\epsilon_{pg}\epsilon^{\alpha\epsilon} - \delta_g^\epsilon\delta_p^\alpha)\epsilon^{\mu\nu}[\epsilon^{a\rho}\gamma_{\mu\nu,a\rho\alpha\epsilon} - \frac{1}{2}\epsilon^{a\rho}\epsilon_{\mu\nu}\gamma_{,a\rho\alpha\epsilon}]$$

$$= 2\lambda(\epsilon_{pg}\epsilon^{\alpha\epsilon} - \delta_g^\epsilon\delta_p^\alpha)[\epsilon^{a\rho}\gamma_{,a\rho\alpha\epsilon} - 2\epsilon^{a\rho}\gamma_{,a\rho\alpha\epsilon}]$$

$$= -2\lambda(\epsilon_{pg}\epsilon^{\alpha\epsilon} - \delta_g^\epsilon\delta_p^\alpha)[\epsilon^{a\rho}\gamma_{,a\rho\alpha\epsilon}]$$

$$= -2\lambda\epsilon_{pg}\epsilon^{\alpha\epsilon}\epsilon^{a\rho}\gamma_{,a\rho\alpha\epsilon} + 2\lambda\epsilon^{a\rho}\gamma_{,a\rho pg}.$$

$$Q_{pg}(R_{ij}R^{ij}) = 2\cdot\frac{\lambda}{2}(\epsilon^{\alpha\epsilon}\delta_p^\mu\delta_g^\nu + \epsilon_{pg}\epsilon^{\mu\epsilon}\epsilon^{\nu\alpha} - \delta_p^\alpha\delta_g^\nu\epsilon^{\epsilon\mu} - \epsilon^{\nu\epsilon}\delta_g^\alpha\delta_p^\mu)(\epsilon^{a\rho}\gamma_{\mu\nu,a\rho\alpha\epsilon} - \frac{1}{2}\epsilon^{a\rho}\epsilon_{\mu\nu}\gamma_{,a\rho\alpha\epsilon})$$

$$= \lambda[\epsilon^{\alpha\epsilon}\epsilon^{a\rho}\gamma_{pg,a\rho\alpha\epsilon} + \epsilon_{pg}\epsilon^{a\rho}\epsilon^{\mu\epsilon}\epsilon^{\nu\alpha}\gamma_{\mu\nu,\alpha a\rho\epsilon}$$

$$-\epsilon^{\alpha\beta}\epsilon^{\mu\epsilon}\gamma_{;q\mu,\epsilon\alpha\beta p}-\epsilon^{\alpha\beta}\epsilon^{\nu\epsilon}\gamma_{;p\nu,\epsilon\alpha\beta q}$$

$$-\tfrac{1}{2}(\epsilon^{\alpha\epsilon}\epsilon^{\alpha\beta}\epsilon_{pq}\gamma_{;,\alpha\beta\alpha\epsilon}+\epsilon_{pq}\epsilon^{\alpha\beta}\epsilon^{\alpha\epsilon}\gamma_{;,\alpha\beta\alpha\epsilon}$$

$$-\epsilon^{\alpha\beta}\gamma_{;,\alpha\beta pq}-\epsilon^{\alpha\beta}\gamma_{;,\alpha\beta pq})]$$

$$=\lambda[\epsilon^{\alpha\epsilon}\epsilon^{\alpha\beta}\gamma_{;pq,\alpha\beta\alpha\epsilon}+\epsilon_{pq}\epsilon^{\alpha\beta}\epsilon^{\mu\epsilon}\sigma_{\mu,\alpha\beta\epsilon}$$

$$-\epsilon^{\alpha\beta}\sigma_{q,\alpha\beta p}-\epsilon^{\alpha\beta}\sigma_{p,\alpha\beta q}-\epsilon_{pq}\epsilon^{\alpha\beta}\epsilon^{\alpha\epsilon}\gamma_{;,\alpha\beta\alpha\epsilon}$$

$$+\epsilon^{\alpha\beta}\gamma_{;,\alpha\beta pq}].$$

In a coordinate system in which $\sigma_\mu = 0$ we get

$$Q_{pq}(R_{ij}R^{ij})=\lambda[\epsilon^{\alpha\epsilon}\epsilon^{\alpha\beta}\gamma_{;pq,\alpha\beta\alpha\epsilon}-\epsilon_{pq}\epsilon^{\alpha\beta}\epsilon^{\alpha\epsilon}\gamma_{;,\alpha\beta\alpha\epsilon}$$

$$+\epsilon^{\alpha\beta}\gamma_{;,\alpha\beta pq}].$$

APPENDIX E

To show that $\sqrt{-g}\,T^{\alpha\beta}{}_{;\beta}=0$ leads to the equation

$$\frac{\partial}{\partial x^\beta}\sqrt{-g}\,T^{\alpha\beta}=-\sqrt{-g}\left\{{\alpha \atop \mu\beta}\right\}T^{\mu\beta}.$$

From the general formula for the covariant derivative of a tensor of the second rank,

$$T^{\alpha\beta}{}_{;\gamma}=T^{\alpha\beta}{}_{,\gamma}+\left\{{\alpha \atop \mu\gamma}\right\}T^{\mu\beta}+\left\{{\beta \atop \mu\gamma}\right\}T^{\alpha\mu}.$$

Therefore

$$\begin{aligned}0=T^{\alpha\beta}{}_{;\beta}&=T^{\alpha\beta}{}_{,\beta}+\left\{{\alpha \atop \mu\beta}\right\}T^{\mu\beta}+\left\{{\beta \atop \mu\beta}\right\}T^{\alpha\mu}\\&=T^{\alpha\beta}{}_{,\beta}+T^{\alpha\beta}\frac{\partial}{\partial x^\beta}(\ln\sqrt{-g})+\left\{{\alpha \atop \mu\beta}\right\}T^{\mu\beta}\\&=\frac{1}{\sqrt{-g}}\frac{\partial}{\partial x^\beta}(\sqrt{-g}\,T^{\alpha\beta})+\left\{{\alpha \atop \mu\beta}\right\}T^{\mu\beta}.\end{aligned}$$

The result follows.

APPENDIX F

To show that in the static case, the 44 component of $T^{\alpha\beta}$ is the density of a point particle.

In this case, $\frac{dz^\alpha}{ds} = 0$ $(\alpha = 1, 2, 3)$ and $\frac{dz^4}{ds} = 1$.

Then

$$T^{44} = \frac{1}{\sqrt{-g}} m \int \frac{dz^4}{ds}\frac{dz^4}{ds} \delta(x^4 - z^4(s)) \delta^3(\underline{x} - \underline{z}(s)) ds$$

$$= \frac{1}{\sqrt{-g}} m \int \delta(x^4 - z^4) \delta^3(\underline{x} - \underline{z}) \frac{ds}{dz^4} dz^4$$

$$= \frac{1}{\sqrt{-g}} m \delta^3(\underline{x} - \underline{z}).$$

To show that this corresponds to the density of a point particle, integrate over a volume surrounding the particle in three-dimensional space to get m.

$$\int \frac{m}{\sqrt{-g}} \delta^3(\underline{x} - \underline{z}) \cdot \sqrt{-g}\, d^3x = m.$$

APPENDIX G

To show that as a $\rightarrow 0$, equation (4) approaches Einstein's equation if we take $\eta = 16\pi G$.

Equation (4) reads

$$-\lambda \epsilon^{a\rho} \gamma_{;pq,a\rho} - 2a\lambda \epsilon^{\alpha\epsilon} \epsilon^{a\rho} \gamma_{;pq,a\rho\alpha\epsilon} = \eta T_{pq}$$

As a $\rightarrow 0$ this becomes

$$-\lambda \epsilon^{a\rho} \gamma_{;pq,a\rho} = \eta T_{pq}$$

Einstein's equation reads (Bergmann (6), p.184)

$$-\lambda \epsilon^{\rho\sigma} \gamma_{;\mu\nu,\rho\sigma} = 16\pi G T_{\mu\nu}$$

Therefore, as a $\rightarrow$ 0, equation (4) approaches that of Einstein if we take $\eta = 16\pi G$.

APPENDIX H

To show that as $b \to 0$,

$$\gamma^{\mu\nu}(x_p) = \frac{-\eta}{\lambda}\left(\frac{m}{b}\right)\int_{-\infty}^{\infty} ds \cdot \frac{1}{\sqrt{-g}} \frac{J_1(u(s)/b)}{u(s)} \dot{z}^{\mu}(s)\dot{z}^{\nu}(s)$$

approaches

$$\frac{-\eta}{\lambda} \frac{m' \dot{z}^{\mu}(s_0)\dot{z}^{\nu}(s_0)}{K\sqrt{-g}\,|_{s_0}}$$

, where all quantities are as defined in the text.

$J_1\left(\frac{u}{b}\right)/u$ is a strongly oscillating function whose oscillations die down as b is decreased. Its value at $u = 0$ is $1/2b$. Its first zero is at $u = \alpha_1 b$ where α_1 is the first zero of $J_1(x)$.

$\alpha_1 b/2b = \alpha_1/2 =$ constant. So as b decreases to 0, we get a delta-function. The value of the constant multiplying the delta-function is found to be

$$\int_{-\infty}^{\infty} \frac{J_1\left(\frac{u}{b}\right)}{u} du = \int_{-\infty}^{\infty} \frac{J_1(x)}{x} dx = 2\int_{0}^{\infty} \frac{J_1(x)}{x} dx = 2.$$

Therefore the limit as $b \to 0$ of $J_1(u/b)/u$ is $2\delta(u^2)$. The argument of the delta-function must be u^2 since $J_1(u)$ is an odd function of u.

Then

$$\gamma^{\mu\nu}(x_p) \to \frac{-2\eta}{\lambda} m' \int_{-\infty}^{\infty} \frac{\delta(u^2)}{\sqrt{-g}} \dot{z}^{\mu}(s)\dot{z}^{\nu}(s)\, ds$$

We use the well-known relation

$$\delta[\phi(x)] = \sum_i \frac{1}{|\phi'(\alpha_i)|} \delta(x - \alpha_i)$$

where $\phi(\alpha_i) = 0$.

In our case $\phi = u^2 = (u_\alpha u^\alpha)$ where $u_\alpha = x_\alpha - z_\alpha$.

Then $\delta(u^2) = \delta(s-s_0) \Big/ \left| \frac{d(u^2)}{ds}\Big|_{s_0} \right|$

where $u^2(s_0) = 0$.

$$d(u^2)/ds = \frac{d\,(u_\alpha u^\alpha)}{ds} = u_\alpha \frac{du^\alpha}{ds} + u^\alpha \frac{du_\alpha}{ds} = 2u_\alpha \frac{du^\alpha}{ds}$$

Therefore

$$\delta(u^2) = \delta(s-s_0) \Big/ \left| 2u_\alpha(s_0) \frac{du^\alpha(s_0)}{ds} \right| .$$

Since s_0 is the point on the world line such that $u_\alpha u^\alpha = 0$ and $u_4 > 0$, $u_\alpha v^\alpha$ is, in the static case, u_4, which is positive. Therefore $u_\alpha v^\alpha$ is positive in all cases, and we can write $\delta(u^2) = \frac{\delta(s-s_0)}{2K}$.

So, $$\gamma^{\mu\nu}(x_\rho) \rightarrow -\frac{2\eta}{\lambda} \frac{m'}{2K|_{s_0}} \int_{-\infty}^{\infty} \frac{\delta(s-s_0)\, \dot{z}^\mu(s)\, \dot{z}^\nu(s)\, ds}{\sqrt{-g}}$$

$$= -\frac{\eta m'}{\lambda} \frac{\dot{z}^\mu(s_0)\, \dot{z}^\nu(s_0)}{K\sqrt{-g}\,|_{s_0}} .$$

APPENDIX I

To show that the Christoffel symbols derived from expression (7) for $\gamma^{\mu\nu}_{,}(x_\rho)$ are as exhibited in equation (8).

We can write

$$\gamma_{,\mu\nu} = \frac{2B}{\lambda} \cdot \frac{\dot{z}_\mu(s_0)\dot{z}_\nu(s_0)}{K}$$

where B = -2Gm.

Then $\gamma_{,\mu\nu,\alpha} = \frac{2B}{\lambda K^2}\{K(v_\mu v_\nu)_{,\alpha} - v_\mu v_\nu (K)_{,\alpha}\}$

$$(v_\mu v_\nu)_{,\alpha} = v_{\mu,\alpha} v_\nu + v_{\nu,\alpha} v_\mu$$
$$= (\partial v_\mu/\partial s_0)(\partial s_0/\partial x^\alpha)v_\nu + (\partial v_\nu/\partial s_0)(\partial s_0/\partial x^\alpha)v_\mu.$$

To find $\partial s_0/\partial x^\alpha$, use $u_\mu u^\mu = 0$. Therefore $(\partial u_\mu/\partial x^\alpha)u^\mu = 0$.

$u_\mu = x_\mu - z_\mu$; $\partial u_\mu/\partial x^\alpha = \epsilon_{\mu\alpha} - \frac{\partial z_\mu}{\partial s_0}\frac{\partial s_0}{\partial x^\alpha}$

Therefore $u^\mu \partial u_\mu/\partial x^\alpha = 0 = u_\alpha - u^\mu v_\mu \partial s_0/\partial x^\alpha$

and $\partial s_0/\partial x^\alpha = u_\alpha/K$.

So

$$(v_\mu v_\nu)_{,\alpha} = \frac{\dot{v}_\mu u_\alpha v_\nu}{K} + \frac{\dot{v}_\nu u_\alpha v_\mu}{K}.$$

Also

$$(K)_{,\alpha} = (u_\mu v^\mu)_{,\alpha} = u_{\mu,\alpha} v^\mu + u_\mu v^\mu_{,\alpha}$$
$$= \left(\epsilon_{\mu\alpha} - \frac{v_\mu u^\alpha}{K}\right)v^\mu + \frac{u_\mu \dot{v}^\mu u_\alpha}{K}$$
$$= v_\alpha - v_\mu v^\mu \frac{u_\alpha}{K} + u_\mu \dot{v}^\mu \frac{u_\alpha}{K}$$

Now

$$ds^2 = \epsilon_{\mu\nu}\, dx^\mu dx^\nu$$

$$1 = \epsilon_{\mu\nu} \frac{dx^\mu}{ds} \frac{dx^\nu}{ds} = \epsilon_{\mu\nu} v^\mu v^\nu = v_\mu v^\mu$$

So, letting $\kappa' = u_\mu \dot{v}^\mu$,

$$(\kappa)_{,\alpha} = v_\alpha - \frac{u_\alpha}{\kappa}(1-\kappa').$$

So

$$\gamma_{\mu\nu,\alpha} = \frac{2B}{\lambda}\left[\frac{\kappa}{\kappa^2}\left(\frac{\dot{v}_\mu u_\alpha v_\nu}{\kappa} + \frac{\dot{v}_\nu u_\alpha v_\mu}{\kappa}\right) - \frac{v_\mu v_\nu}{\kappa^2}\left(v_\alpha - \frac{u_\alpha}{\kappa}(1-\kappa')\right)\right]$$

$$= \frac{2B}{\lambda}\left[\frac{u_\alpha}{\kappa^2}(\dot{v}_\mu v_\nu + \dot{v}_\nu v_\mu) - \frac{v_\mu v_\alpha v_\nu}{\kappa^2} + \frac{v_\mu v_\nu u_\alpha (1-\kappa')}{\kappa^3}\right].$$

So we have $\gamma_{\alpha\nu,\mu} + \gamma_{\mu\alpha,\nu} - \gamma_{\mu\nu,\alpha}$

$$= \frac{2B}{\lambda}\Big[\frac{1}{\kappa^2}\Big\{u_\mu \dot{v}_\alpha v_\nu + u_\mu \dot{v}_\nu v_\alpha + u_\nu \dot{v}_\mu v_\alpha + u_\nu \dot{v}_\alpha v_\mu$$
$$- u_\alpha \dot{v}_\mu v_\nu - u_\alpha \dot{v}_\nu v_\mu - v_\alpha v_\nu v_\mu\Big\}$$
$$+ \frac{1-\kappa'}{\kappa^3}\Big\{v_\alpha v_\nu u_\mu + v_\mu v_\alpha u_\nu - v_\mu v_\nu u_\alpha\Big\}\Big].$$

Again,

$$\epsilon_{\alpha\nu}\gamma_{,\mu} = \epsilon_{\alpha\nu}\epsilon^{pq}\gamma_{pq,\mu}$$

$$= \epsilon_{\alpha\nu}\epsilon^{pq}(2B/\lambda)\left[\frac{u_\mu}{\kappa^2}(\dot{v}_p v_q + \dot{v}_q v_p) - \frac{v_p v_q v_\mu}{\kappa^2} + v_p v_q u_\mu \frac{(1-\kappa')}{\kappa^3}\right]$$

and

$$\epsilon^{pq}(\dot{v}_p v_q + \dot{v}_q v_p) = \dot{v}_p v^p + \dot{v}_q v^q = 2\dot{v}_p v^p$$

$$= \frac{\partial(v_p v^p)}{\partial s_0} = \frac{\partial(1)}{\partial s_0} = 0.$$

Using

$$\epsilon^{pq} v_p v_q = v_p v^p = 1$$

we have

$$\epsilon_{a\nu}\gamma_{,\mu} = \epsilon_{a\nu}(2B/\lambda)[-v_\mu/K^2 + u_\mu(1-K')/K^3].$$

Then

$$-\epsilon_{a\nu}\gamma_{,\mu} - \epsilon_{\mu a}\gamma_{,\nu} + \epsilon_{\mu\nu}\gamma_{,a}$$

$$= \frac{2B}{\lambda}\Big\{\frac{1-K'}{K^3}(\epsilon_{\mu\nu}u_a - \epsilon_{\mu a}u_\nu - \epsilon_{a\nu}u_\mu)$$

$$-\frac{1}{K^2}(\epsilon_{\mu\nu}v_a - \epsilon_{\mu a}v_\nu - \epsilon_{a\nu}v_\mu)\Big\}.$$

Therefore

$$\left\{{\rho \atop \mu\nu}\right\} = \frac{\lambda}{2}\epsilon^{a\rho}(h_{a\nu,\mu} + h_{\mu a,\nu} - h_{\mu\nu,a})$$

$$= \frac{\lambda}{2}\epsilon^{a\rho}(\gamma_{a\nu,\mu} + \gamma_{\mu a,\nu} - \gamma_{\mu\nu,a} - \frac{1}{2}\epsilon_{a\nu}\gamma_{,\mu}$$

$$-\frac{1}{2}\epsilon_{\mu a}\gamma_{,\nu} + \frac{1}{2}\epsilon_{\mu\nu}\gamma_{,a})$$

$$= B\epsilon^{a\rho}\Big[\frac{1}{K^2}\{u_\mu \dot{v}_a v_\nu + u_\mu \dot{v}_\nu v_a + u_\nu \dot{v}_\mu v_a + u_\nu \dot{v}_a v_\mu$$

$$- u_a \dot{v}_\mu v_\nu - u_a \dot{v}_\nu v_\mu - v_a v_\nu v_\mu\}$$

$$+\frac{(1-K')}{K^3}\{v_a v_\nu u_\mu + v_\mu v_a u_\nu - v_\mu v_\nu u_a\}$$

$$+\frac{1}{2}\frac{(1-K')}{K^3}(\epsilon_{\mu\nu}u_a - \epsilon_{\mu a}u_\nu - \epsilon_{a\nu}u_\mu)$$

$$-\frac{1}{2K^2}(\epsilon_{\mu\nu}v_a - \epsilon_{\mu a}v_\nu - \epsilon_{a\nu}v_\mu)\Big]$$

which, on carrying out the contractions with respect to the index a, reduces to the expression given by equation (8).

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